MATHEMATICS

# ON THE ALMOST PERIODICITY OF TRIGONOMETRIC POLYNOMIALS IN CONSTRUCTIVE MATHEMATICS

BY

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In the following we shall accept the definition of almost periodicity given originally by H. Bohr. Let $f: \mathbf{R} \to \mathbf{C}$ be a map. For any positive real $\varepsilon$ an *$\varepsilon$-translation number* of $f$ will mean a real $\tau$ with the property that

$$|f(x+\tau)-f(x)| \leqslant \varepsilon$$

for all real $x$. A set $T$ of reals will be called *relatively dense* when it is possible to construct a positive real $L$ with the property that every open interval of length $L$ contains a point in $T$. And we call $f$ *almost periodic* when it is continuous, and when for every positive real $\varepsilon$ it is possible to construct a relatively dense set of $\varepsilon$-translation numbers of $f$.

Let us briefly review the salient properties of almost periodic maps. As in classical mathematics it is easily checked that an almost periodic map is bounded, that an almost periodic map is uniformly continuous, and that the limit of a uniformly convergent sequence of almost periodic maps is almost periodic. However, the supremum axiom for real numbers is not accepted in constructive mathematics, so that it cannot be concluded, without further proof, that an almost periodic map $f$ necessarily has a supremum. The gap can be bridged by the following device. For any positive integer $n$ we denote by $s_n$ the supremum of $|f|$ on $[-n, n]$. Given any positive real $\varepsilon$ it will be possible to indicate a positive integer $N$ such that every open interval of length $N$ contains an $\varepsilon$-translation number of $f$. I claim that $|s_n - s_N| \leqslant \varepsilon$ for all $n \geqslant N$. It will suffice to show that for any point $y$ there is a point $x$ with $-N \leqslant x \leqslant N$ for which

$$\big|\, |f(y)| - |f(x)| \,\big| \leqslant |f(x)-f(y)| \leqslant \varepsilon:$$

that can be accomplished by choosing an $\varepsilon$-translation number $\tau$ with $|y-\tau| \leqslant N/2$ and taking $x$ to be $y-\tau$. We conclude that $(s_n)$ converges, so that $s = \operatorname{limit}_{n\to\infty} s_n$ will be the supremum of $f$.

Furthermore it can be established in exactly the same way as in the classical theory (see for instance the book by Bohr) that every almost periodic map $f$ has a mean value

$$M(f) = \operatorname*{limit}_{T\to\infty} \frac{1}{T} \int_0^T f(x)\,dx.$$

The peculiarities of almost periodic maps in constructive mathematics can be illustrated by considering maps of the form $e^{2\pi i\lambda x}$. I claim that such a map is almost periodic if and only if $\lambda=0$ or $\lambda\neq 0$. The condition is certainly sufficient since then the map is periodic. If $e^{2\pi i\lambda x}$ is almost periodic it will have a mean value whose modulus will be positive or $<1$. In the latter case it is an easy matter to show that $\lambda\neq 0$; and in the former case the assumption that $\lambda\neq 0$ leads immediately to a contradiction so that $\lambda=0$.

The closure properties of almost periodic maps are profoundly affected by this peculiarity. Certainly the almost periodic maps are closed under scalar multiplication, translation, conjugation, the operation of taking the modulus and the operation of taking the square. On the other hand they are *not* closed under multiplication, as is now evident by considering the maps $e^{2\pi i\lambda x}$ with $\lambda=0$ or $\lambda\neq 0$. And in view of the relation,

$$4fg=(f+g)^2-(f-g)^2$$

they are *not* closed under addition.

We fix our notation by agreeing that a *trigonometric polynomial* $P$ will mean a map $P:\mathbf{R}\to\mathbf{C}$ defined by a formula

$$P(x)=\sum_{k=1}^{\infty} c_k e^{2\pi i\lambda_k x} \quad (x\in\mathbf{R})$$

where each $c_k$ is a complex number which is either $\neq 0$ or $=0$, and each $\lambda_k$ is a real number. It will be assumed that the *exponents*, i.e. the $\lambda_k$ for which $c_k\neq 0$, are non-zero and distinct. Also, it will be assumed that the convergence is uniform. It is to be borne in mind that it may not be possible to compute the cardinal of the set of exponents. The considerations above make it clear that a trigonometric polynomial is not automatically almost periodic, and raise the problem of finding necessary and sufficient conditions for this to be the case. Recall at this point that a set of reals $X$ is called *discrete* when any two elements are either equal or apart. It turns out that the discreteness of the exponents does not ensure almost periodicity: we shall therefore introduce a stronger notion. We call $X$ *rationally discrete* when the set of all finite linear combinations $\sum_{k=1}^{n} r_k x_k$, with $r_1, \ldots, r_n$ rational and $x_1, \ldots, x_n$ in $X$, is discrete. Our result can now be stated in the following form.

*A necessary and sufficient condition for a trigonometric polynomial to be almost periodic is that its exponents are rationally discrete.*

To this end we introduce the following definition. The real numbers $\mu_1, \ldots, \mu_n$ are said to be *phased* when for every positive real number $\varepsilon$ it is possible to construct a relatively dense set $T$ of reals with the property that for any $\tau$ in $T$ there are integers $p_1, \ldots, p_n$ such that

$$|\mu_k\tau-p_k|\leqslant\varepsilon \quad (1\leqslant k\leqslant n).$$

I. *A sufficient condition for a trigonometric polynomial $P$ to be almost periodic is that any finite number of its exponents are phased.*

Since $P$ is a uniform limit of a sequence of finite trigonometric polynomials it will suffice to establish the result in the case when $P$ is finite. We write $P(x) = \sum_{k=1}^{n} c_k e^{2\pi i\lambda_k x}$ and may as well suppose that all the coefficients are $\neq 0$. Choose $\varepsilon > 0$. Each of the maps $c_k e^{2\pi i\lambda_k x}$ is uniformly continuous, so it will be possible to determine a positive real $\delta$ which is a common modulus of uniform continuity for these maps corresponding to $\varepsilon/n$. Assuming the exponents to be phased we know that there is a relatively dense set of reals $T$ with the property that for any $\tau$ in $T$ there are integers $p_1, \ldots, p_n$ such that

$$|\lambda_k\tau - p_k| \leqslant \delta\Lambda \quad (1 \leqslant k \leqslant n)$$

where

$$\Lambda = \min\{|\lambda_1|, \ldots, |\lambda_n|\}.$$

Such a $\tau$ lies within a distance $\delta$ of a period of each of the maps $c_k e^{2\pi i\lambda_k x}$ and is thus a common $\varepsilon/n$-translation number of these maps. It follows that $\tau$ is an $\varepsilon$-translation number for $P$; and that establishes the almost periodicity of $P$.

II. *A necessary condition for a trigonometric polynomial $P$ to be almost periodic is that any finite number of its exponents are phased.*

Let $P(x) = \sum_{k=1}^{\infty} c_k e^{2\pi i\lambda_k x}$ and consider any finite number $\lambda_{k_1}, \ldots, \lambda_{k_m}$ of its exponents. Choose any positive real $\varepsilon$. Assuming $P$ to be almost periodic it will be possible to construct a relatively dense set of reals $\tau$ with the property that

$$|P(x+\tau) - P(x)|^2 \leqslant \varepsilon^2$$

for all real $x$. Now the left hand side of this inequality defines an almost periodic map (for any real $\tau$) and so has a mean value for which

$$M\{|P(x+\tau) - P(x)|^2\} \leqslant \varepsilon^2.$$

This we re-write as

$$M\{|\sum_{k=1}^{\infty} C_k e^{2\pi i\lambda_k x}|^2\} \leqslant \varepsilon^2$$

where

$$C_k = c_k\{e^{2\pi i\lambda_k\tau} - 1\}.$$

Calculation reduces this to

$$\sum_{k=1}^{\infty} |C_k|^2 \leqslant \varepsilon^2$$

yielding the inequalities

$$|C_k| \leqslant \varepsilon$$

and in particular

$$|c^{2\pi i\lambda_{k_j}\tau} - 1| \leqslant \varepsilon/c \quad (1 \leqslant j \leqslant m)$$

where

$$c = \min\{|c_{k_1}|, \ldots, |c_{k_m}|\}.$$

Thus, provided $\varepsilon$ is small enough, there will be integers $p_1, \ldots, p_m$ such that

$$|\lambda_{k_j}\tau - p_j| \leqslant \pi\varepsilon/2c \quad (1 \leqslant j \leqslant m).$$

And since these last inequalities hold for a relatively dense set of $\tau$ we conclude that $\lambda_{k_1}, \ldots, \lambda_{k_m}$ are indeed phased.

To simplify further progress we prove a straightforward combinatorial lemma.

*If $\lambda_1, \ldots, \lambda_n$ are phased and $\lambda_{n+1}$ is linearly dependent on $\lambda_1, \ldots, \lambda_n$ over the rationals then $\lambda_1, \ldots, \lambda_n, \lambda_{n+1}$ are phased.*

There will be a positive integer $t$ and integers $s_1, \ldots, s_n$ such that

$$t\lambda_{n+1} = s_1\lambda_1 + \ldots + s_n\lambda_n.$$

Let $\delta$ be a positive real. Since $\lambda_1, \ldots, \lambda_n$ are phased it will be possible to find a positive real $L$ such that in any open interval of length $L$ there is a point $\tau$ with the property that there are integers $p_1, \ldots, p_n$ for which

$$|\lambda_k\tau - p_k| \leqslant \frac{\delta}{|s_1| + \ldots + |s_n| + |t|} \qquad (1 \leqslant k \leqslant n).$$

Consider now any interval $]ta, tb[$ of length $tL$. In the interval $]a, b[$ there will be a point $\tau$ with the properties just mentioned. Now $t\tau$ lies in $]ta, tb[$, and

$$|\lambda_k t\tau - tp_k| \leqslant \delta \qquad (1 \leqslant k \leqslant n).$$

Moreover, calculation yields

$$|\lambda_{n+1} t\tau - \sum_{k=1}^{n} p_k s_k| \leqslant \delta.$$

And from these inequalities we deduce the result.

III. *A sufficient condition for the reals $\lambda_1, \ldots, \lambda_n$ to be rationally discrete is that they are phased.*

Consider first the case when $n = 1$. We suppose that $\lambda_1$ is phased: it will suffice to show that $\lambda_1 = 0$ or $\lambda_1 \neq 0$. It will be possible to find a positive real $L$ such that for any positive integer $k$ there is a point $\tau_k$ with $(k-1)L < \tau_k < kL$ which has the property that there is an integer $p_k$ for which

$$|\lambda_1 \tau_k - p_k| < \tfrac{1}{4}.$$

Thus

$$|p_{k+1} - p_k| \leqslant |p_{k+1} - \lambda_1\tau_{k+1}| + |\lambda_1\tau_{k+1} - \lambda_1\tau_k| + |\lambda_1\tau_k - p_k| < \tfrac{1}{4} + 2L|\lambda_1| + \tfrac{1}{4}.$$

Now $\lambda_1 \neq 0$ or $|\lambda_1| < \tfrac{1}{4}L$. In the former case there is nothing further to prove. In the latter case we deduce that $p_{k+1} = p_k$ for all $k$. If $p_1 \neq 0$ then clearly $\lambda_1 \neq 0$. If $p_1 = 0$ then $|\lambda_1\tau_k| < \tfrac{1}{4}$ for all $k$: but then the assumption $\lambda_1 \neq 0$ leads to a contradiction as $\tau_k$ can be made as large as we please— and we conclude that $\lambda_1 = 0$.

Consider next the general case when $\lambda_1, \ldots, \lambda_n$ are phased. If will suffice to show that any finite rational linear combination $\lambda_{n+1} = \sum_{k=1}^{n} r_k \lambda_k$ is either $=0$ or $\neq 0$. By the lemma above $\lambda_1, \ldots, \lambda_n, \lambda_{n+1}$ will be phased, so $\lambda_{n+1}$ will be phased, and hence (by the case $n=1$) $=0$ or $\neq 0$.

IV. *A necessary condition for the reals $\lambda_1, \ldots, \lambda_n$ to be rationally discrete is that they are phased.*

To establish this result we shall need a constructive variant of a theorem due originally to Kronecker. Our proof has been based on that given by ESTERMANN in [3].

*Let $\theta_1, \ldots, \theta_n$, 1 be rationally discrete real numbers, let $x_1, \ldots, x_n$ be any real numbers and let $\varepsilon > 0$: either $\theta_1, \ldots, \theta_n$, 1 are linearly dependent over the rationals, or it is possible to construct integers $N$, $v_1, \ldots, v_n$ such that*

$$|N\theta_k - v_k - x_k| \leqslant \varepsilon \qquad (1 \leqslant k \leqslant n).$$

As a preliminary we establish the existence of integers $s, t_1, \ldots, t_n$ with $s > 0$ for which

$$|s\theta_k - t_k| \leqslant \varepsilon/2 \qquad (1 \leqslant k \leqslant n).$$

Observe first that if $\theta$ is a real for which $\theta$, 1 are rationally discrete it will be possible to define the integral part of $\theta$, and hence the fractional part of $\theta$. For any integer $u$ we denote by $\Theta_u$ the vector in $\mathbf{R}^n$ whose $k^{\text{th}}$ component is the fractional part of $u\theta_k$. It will suffice to produce two $\Theta_u$, with different indices, which differ by $\leqslant \varepsilon/2$ in the usual supremum norm on $\mathbf{R}^n$. All the $\Theta_u$ lie in the unit cube $X$ in $\mathbf{R}^n$. $X$ is compact so it can be covered by finitely many subsets $x_1, \ldots, x_m$ which are all compact and of diameter $\leqslant \varepsilon/2$: this proposition can be found on p. 100 of the book by Bishop. We consider the $\Theta_u$ with $1 \leqslant u \leqslant m+1$. Since $\theta_1, \ldots, \theta_n$, 1 are rationally discrete any two of the $\Theta_u$ are either equal or lie apart. Hence there is at least one equality amongst the $\Theta_u$, or they are distinct, in which case it will be possible to indicate two which lie in the same $X_k$. And in either case we have attained our object.

Now we follow Estermann closely. We suppose that $n > 1$ and that the theorem holds for $n-1$. For each $k$ with $1 \leqslant k \leqslant n$ we can decide whether or not $\theta_k$ equals $t_k/s$. In the event that at least one such equality occurs $\theta_1, \ldots, \theta_n$, 1 will be linearly dependent over the rationals. Otherwise we introduce

$$\phi_k = \frac{s\theta_k - t_k}{s\theta_n - t_n} \qquad (1 \leqslant k \leqslant n).$$

It is evident that $\phi_1, \ldots, \phi_{n-1}$, 1 are rationally discrete. By the induction hypothesis there are two possibilities. The first is that $\phi_1, \ldots, \phi_{n-1}$, 1 are linearly dependent—in which case also $\theta_1, \ldots, \theta_n$, 1 will be linearly dependent. The second is that there will be integers denoted by $w_n, w_1, \ldots, w_{n-1}$ such that

$$|w_n\phi_k - w_k - (x_k - x_n\phi_k)| \leqslant \varepsilon/2 \qquad (1 \leqslant k \leqslant n).$$

If we choose an integer $w$ such that

$$\left| w - \frac{w_n + x_n}{s\theta_n - t_n} \right| < 1$$

and put

$$N = ws:\, v_k = w_n t_k + w_k$$

then

$$|N\theta_k - v_k - x_k| \leqslant \varepsilon \qquad (1 \leqslant k \leqslant n)$$

as required.

Although it is not strictly relevant to our problem it seems worthwhile pointing out that since linearly independent reals are automatically rationally discrete we can deduce immediately the classical result of Kronecker.

*Let $\theta_1, \ldots, \theta_n, 1$ be linearly independent over the rationals, let $x_1, \ldots, x_n$ be any reals and let $\varepsilon > 0$: there are integers $N, v_1, \ldots, v_n$ such that*

$$|N\theta_k - v_k - x_k| \leqslant \varepsilon \qquad (1 \leqslant k \leqslant n).$$

We can now proceed with the proof of IV, which is by induction on $n$. The case when $n = 1$ is trivial. Suppose that the theorem holds for $n$, and let $\lambda_1, \ldots, \lambda_n, \lambda_{n+1}$ be rationally discrete. We remark that at any point in the proof the possibility that $\lambda_1, \ldots, \lambda_n, \lambda_{n+1}$ are linearly dependent will lead to the result: in such a case one $\lambda_k$ will be a finite rational linear combination of the remaining $n$, these will be phased in view of the induction hypothesis, and the result will be immediate from an earlier lemma. Bearing this remark in mind it will be no restriction to suppose that $\lambda_{n+1} > \lambda_n > \ldots > \lambda_1 > 0$. Determine a positive integer $t$ such that $t > \lambda_{n+1}/\lambda_1$ and choose $\varepsilon > 0$. Let $x_1, \ldots, x_m$ be an $\varepsilon\lambda_1/2t$-net in the unit cube in $\mathbf{R}^n$ where

$$x_j = (x_{j,1}, \ldots, x_{j,n}) \qquad (1 \leqslant j \leqslant m).$$

Now $1, \lambda_2/\lambda_1, \ldots, \lambda_{n+1}/\lambda_1$ are rationally discrete, so our variant on Kronecker's theorem yields two possibilities. The first is that they are linearly dependent, in which case $\lambda_1, \ldots, \lambda_n, \lambda_{n+1}$ are also linearly dependent, and the result follows. The second is that for each $j$ with $1 \leqslant j \leqslant m$ there are integers

$$N_j,\, v_{j,1},\, \ldots,\, v_{j,n}$$

such that

$$\left| N_j \frac{\lambda_k}{\lambda_1} - v_{j,k} - x_{j,k-1} \right| \leqslant \varepsilon\lambda_1/2t$$

for $2 \leqslant k \leqslant n+1$. Put

$$N = \max\{|N_1|, \ldots, |N_m|\}.$$

I assert that in every open interval of length $tN/\lambda_1$ centred at an integral multiple of $1/\lambda_1$ there is a point $\tau$ which lies within a distance $\varepsilon$ of an integral multiple of $1/\lambda_k$ for $2 \leqslant k \leqslant n+1$: and that will establish that

$\lambda_1, \ldots, \lambda_n, \lambda_{n+1}$ are phased. To this end choose any integral multiple $p_1/\lambda_1$ of $1/\lambda_1$, and determine integers $p_2, \ldots, p_{n+1}$ such that

$$0 \leqslant \frac{p_k}{\lambda_k} - \frac{p_1}{\lambda_1} \leqslant \frac{1}{\lambda_1} \qquad (2 \leqslant k \leqslant n+1)$$

yielding

$$0 \leqslant \frac{1}{t}\left(p_k - p_1 \frac{\lambda_k}{\lambda_1}\right) \leqslant 1 \qquad (2 \leqslant k \leqslant n+1).$$

Now compute $J$ such that

$$\left| x_{J,k-1} - \frac{1}{t}\left(p_k - p_1 \frac{\lambda_k}{\lambda_1}\right) \right| \leqslant \frac{\varepsilon \lambda_1}{2t}$$

for $2 \leqslant k \leqslant n+1$: that allows us to write

$$\left| \left(N_J \frac{\lambda_k}{\lambda_1} - v_{J,k}\right) - \frac{1}{t}\left(p_k - p_1 \frac{\lambda_k}{\lambda_1}\right) \right| \leqslant \frac{\varepsilon \lambda_1}{t}$$

for the same values of $k$. And this inequality we can re-write as,

$$\left| (tN_J + p_1)\frac{1}{\lambda_1} - (tv_{J,k} + p_k)\frac{1}{\lambda_k} \right| \leqslant \varepsilon$$

for $2 \leqslant k \leqslant n+1$. We take $\tau = (tN_J + p_1)/\lambda_1$ and observe that

$$\left| \tau - \frac{p_1}{\lambda_1} \right| = |tN_J/\lambda_1| \leqslant tN/\lambda_1.$$

Our main result now follows from I, II, III and IV.

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## REFERENCES

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3. Estermann, T., A Proof of Kronecker's Theorem by Induction, J. London Math. Soc. 8, 18–20 (1933).